USSR Academy of Sciences Proceedings

1989. Volume 308. № 6.

UDC 550.3:517.97

GEOPHYSICS

© Academician A.S. Alekseev, G.N. Erokhin

INTEGRATION IN GEOPHYSICAL INVERSE PROBLEMS

(INTEGRATED GEOPHYSICS)

Inverse problems of single geophysical methods – gravics [1], electric exploration [2], seismic [3-6], methods based on potential geophysical fields [7] – have been intensively investigated over the past three decades in theoretical Geophysics.

However, when pragmatically studying earth constitution and searching for mineral deposits, normally to advance exploration efficiency, a certain set of geophysical methods allowing for a more diversified description of the geological feature rather than an isolated method is used. Nevertheless, if several methods are applied, each of them, in fact, is used independently and doesn't always allow for obtaining sufficient (unambiguous and unalterable) information. Autonomously obtained results of separate methods are interconnected without applying integrated mathematical models and, consequently, the results obtained is integration quantitative evidence which is not Physics and Mathematics level driven.

In papers [8, 9] the authors have attempted to combine the models of different methods within the framework of unified statement of geophysical inverse problems (integral Geophysics). A similar approach is employed in the work [10]. In addition to the theoretical works, the integral Geophysics inverse problems system science approaches for integral information environment formation have appeared recently [11].

All such approaches require integral mathematical models to be formed and theoretically analyzed.

This work shows that if inverse problems that are integrated at the statistical level are applied to co-processing of fields of different geophysical nature, correctness set (guaranteed uniqueness and fixity sets) expands and accuracy of the environment parameter definition increases. The constructive method for solving combined inverse problems based on minimization of a certain integral objective functional is suggested.

Let us consider a set of operator equations

$$(1) \qquad \mathfrak{M}_\nu \mathrm{u}_\nu = \mathrm{f}_\nu(x,t),$$

where $x \in D \subseteq R^n$, $t \in [0, \mathrm{T}]$, $\nu = 1,...,m$, $\mathfrak{M}_\nu$ - is a certain matrix with $\mathrm{N}_\nu \times \mathrm{N}_\nu$ differential operator of order $k_\nu \le 2$. The equation set (1) describes different geophysical fields

$u_\nu = (u_{1\nu},...,u_{N\nu})$, set up by the sources $f_\nu = (f_{1\nu},...,f_{N\nu})$ which probably are of absolutely different physical nature.

Let us assume that $Q = D\times[0,T]$, $q = (x,t)\in Q$ and on ∂Q relevant initial and end conditions are established which ensure uniqueness of solution for a set of direct problems for m equations (1) in certain smooth enough classes of functions.

Assume further that operators $\mathfrak{M}_\nu = \mathfrak{M}_\nu\left(d_\nu(x), D_x, \partial/\partial t\right)$ are known within the accuracy of vector functions $d_\nu(x) = \left(d_{1\nu}(x),...,d_{L\nu}(x)\right)$ which occur at random and follow a multi-dimensional regular distributions which correspond to known vectors of mathematical expectation and covariance matrices and satisfy I - determinate constraint equations of the type $F_i\left(d_\nu(x)\right) = 0, i = 1,...,I$. Let us assume that the connections are of such a nature that allow for distinguishing $L = \sum_{\nu=1}^{m} L_\nu - I$ of independent parameters in mathematical models of methods combined. Furthermore, let's denote these parameters by generalized vector $\underline{\eta}(x) = \left(\eta_1(x),...,\eta_L(x)\right)$ and express the constraint equations as $d_\nu = d_\nu(\underline{\eta})$. Thus, we will assume hereafter that the aprior averages $\underline{\eta}_0(x)$ and covariance matrices $\Gamma_\eta(q,q'), q,q'\in Q$ are known.

The inverse problem of complex quantitative interpretation of geophysical investigations is to find generalized vector of parameters $\underline{\eta}(x)$ of a set of operators $\{\mathfrak{M}_\nu\}, \nu = 1,...,m$ using the information of the type

$$\text{(2)} \qquad v_\nu(r,t) = u_\nu\left(r,t;d_\nu(\underline{\eta})\right) + \underline{\varepsilon}_\nu(r,t)$$

Here $r\in R_\nu \subset D$, R_ν is a certain manifold on which information for ν method is measured, $\varepsilon_\nu(r,t)$ - is a random disturbance which has a regular distribution with zero mean and a known covariance matrix $\Gamma_{\varepsilon_\nu}(q,q')$. It is also assumed that noise ε_ν is statistically independent on ν.

Let us remark here that by the assumed random nature of vector sought and presence of noise $\varepsilon_\nu(r,t)$ in data (2) the solution of the integral inverse problem is to be considered as finding of optimal estimate $\hat{\underline{\eta}}(x)$ for generalized vector of parameters $\underline{\eta}(x)$ [12].

Lemma *1. Let us assume that covariance matrices $\Gamma_{\eta,}\Gamma_{\varepsilon_\nu}, \nu = 1,2,...,m,$ are written as*

$$\Gamma_\eta(q,q') = G_\eta(x)\delta(t)\delta(q-q'), \Gamma_{\varepsilon_\nu}(q)\delta(q-q'),$$

where $G_{\eta,}G_{\varepsilon_\nu}$ - are positively defined symmetric matrices with $L\times L u N_\nu \times N_\nu$ respectively. Then the problem of finding of optimal estimate $\hat{\eta}$ using the method of peak aposterior probability density is equivalent to search for generalized vector of parameters η which minimizes functional.

$$\text{(3)} \qquad J(\eta) = \sum_{\nu=1}^{m}\left\langle\left(v_\nu - u_\nu\left(q;d_\nu(\underline{\eta})\right)\right),\left(v_\nu - u_\nu\left(q;d_\nu(\underline{\eta})\right)\right)\right\rangle_{B1} + \left\langle\left(\underline{\eta}-\underline{\eta}_0\right),\left(\underline{\eta}-\underline{\eta}_0\right)\right\rangle_{B2}.$$

Here $\langle,\rangle_B$ - vector functions dot product in Hilbert weight space H_B:

$$\text{(4)} \qquad \langle a,b\rangle_B = \int_Q a^T(q)B(q)b(q)\,dq,$$

$$(5) \qquad B_1(q) = G_{\varepsilon_\nu}^{-1}\delta_{R\nu},$$

$$(6) \qquad B_2(q) = G_\eta^{-1}(x)\delta(t),$$

and $\delta_{R\nu}$ *- zero order generalized function with support which is the same as variety* R_ν.

The proof is based on Bayesian theorem and use of aprior statistical assumptions on noise and the required parameters of vector $\underline{\eta}$.

Let's look at the adjoint equation

$$(7) \qquad \mathfrak{M}_\nu^* \mathrm{u}_\nu^* = g_\nu(x,t)$$

with zero boundary and initial conditions in $Q = D \times [0,\mathrm{T}]$ (the latter in case of hyperbolic operator $\mathfrak{M}_\nu$ is: $\mathrm{u}_\nu^*\big|_{t=T} = 0, \quad \mathrm{u}_{\nu t}^*\big|_{t=\mathrm{T}} = 0$. We can show that for solving the equation (7) and solving u_ν equation (1) with zero boundary and initial conditions (the latter are standard for hyperbolic operator: $\mathrm{u}_\nu\big|_{t=0} = 0, \mathrm{u}_\nu\big|_{t=0} = 0$) there holds the equality

$$(8) \qquad \langle \mathrm{u}_\nu, g_\nu \rangle_{B_0} = \langle \mathrm{u}_\nu^*, \mathrm{f}_\nu \rangle_{B_0}.$$

Here B_0 - is the dot product identity matrix (4).

In this regard let's note some details of fundamental solution of the adjoint equation (7) in case of the hyperbolic operator $\mathfrak{M}_\nu$. For example, if $\mathfrak{M}_\nu$ - is a wave operator with constant propagation velocity a. Then fundamental solution of the adjoint problem will be

$$(9) \quad \mathcal{E}_+(x,t) = \frac{\theta_+(t-\mathrm{T})\delta(|x| + \theta(t-\mathrm{T}))}{4\pi a|x|}, \quad \theta_+(t) = \{^{1, t\le 0,}_{0, t>0.}$$

Therefore, in the adjoint problem time changes to the side of negative values – from T to 0, whereas for the wave equation having the following fundamental solution

$$\mathcal{E}_-(x,t) = \frac{\theta_-(t)\delta(|x| + \theta(t-T))}{4\pi a|x|}, \theta_+(t) = \begin{cases} 1, t\le 0, \\ 0, t<0, \end{cases}$$

it may change to the side of positive values – from 0 to T.

Lemma 2. Let's assume that $\mathrm{u}_\nu^*(x,t)$ *- is a solution of the adjoint equation (7) having the following right side*

$$(10) \qquad g_\nu(x,t) = \left(\mathrm{v}_\nu(r,t) - \mathrm{u}_\nu\left(r,t;\mathrm{d}_\nu(\underline{\eta})\right)\right) G_{\varepsilon_\nu}^{-1}\delta_{R_\nu}$$

and zero initial-boundary conditions. Then if the inequation works

$$(11) \qquad \sum_{\nu=1}^{m} \int_Q \left\| \left(\frac{\partial \mathrm{u}_\nu}{\partial \underline{\eta}}\right)^{\mathrm{T}} G_{\epsilon_\nu}^{-1} \frac{\partial \mathrm{u}_\nu}{\partial \underline{\eta}} + G_\eta^{-1} \right\| dq \le c_0^2,$$

where c_0 *- is an arbitrary constant and the norm under the integral sign represents Euclidean norm of the matrix with* $L \times L$*, Frechet derivative of the functional (3) exists and is presented as*

$$(12) \qquad J_\eta(\underline{\eta}) = 2\left(\sum_{\nu=1}^{m} \int_0^T \left(\frac{\partial \mathfrak{M}_\nu}{\partial \underline{\eta}}\right)^T \mathrm{u}_\nu \mathrm{u}_\nu^* dt + G_\eta^{-1}(\underline{\eta} - \underline{\eta}_0) \right)$$

The proof is realized using functional perturbation (3). The condition (11) is sufficient for Frechet derivative existence. As for the functional derivative (12), it is determined on the base of the method for equation perturbation (1) using the relation (8).

Remark 1. The problem of adjoint equation solution (7) with the right side (10) in case of hyperbolic operator $\mathfrak{M}_v$ is tightly coupled with the widely known seismic survey problem of wave field continuation [13] and also with optical wavefront conjugation using nonlinear optical media (dynamic holography) [14]. In fact, in conformity with the structure of fundamental solution (9) of the adjoint equation (7), the sources of the kind (10) at time $t = \mathrm{T}$ start emitting information registered as «the last», and at the end of emission moment $(t = 0)$ signals are generated into the medium which are registered as «the first» when being viewed. The adjoint problem solution is widely used in meteorology for problems of natural environment protection [15], where the solution $\mathrm{u}_v^*(x,t)$ is referred to as value function.

Definition 1. We denote $\mathcal{M}_{\eta_o}^c$ set of vector-functions $\underline{\eta}(x)$ so that the inequation (13) holds for them

$$\left\langle \left(\underline{\eta} - \underline{\eta}_0\right), \left(\underline{\eta} - \underline{\eta}_0\right) \right\rangle_{B_2} \le c^2 . \qquad (13)$$

Here scalar product id defined by the formulas (4), (6), $\underline{\eta}_0$ - aprior average, c – some constant. It can be shown that $\mathcal{M}_{\eta_o}^c$ is a convex and feebly compact subset of Hilbert weight space H_{B_2}.

Lemma 3. Let $\mathcal{M}_{\eta_0}^c$ be some fixed set of vector-functions $\underline{\eta}(x)$ of the type (12) for which the conditions of lemmas 1 and 2 are satisfied. Suppose also that for arbitraries $\underline{\eta}^1, \underline{\eta}^2 \in \mathcal{M}_{\eta_0}^c$ the following inequality holds

$$\sum_{v=1}^{m} \int_Q \left(\mathrm{u}_v^{1*} \left(\left(\frac{\partial \mathfrak{M}_v}{\partial \underline{\eta}^1} \right)^{\mathrm{T}} \mathrm{u}_v^1 \right)^{\mathrm{T}} - \mathrm{u}_v^{2*} \left(\left(\frac{\partial \mathfrak{M}_v}{\partial \underline{\eta}^2} \right)^{\mathrm{T}} \mathrm{u}_v^2 \right)^{\mathrm{T}} \right) \left(\underline{\eta}^1 - \underline{\eta}^2 \right) dq + 2c^2 \ge 0 ,$$

where $\mathrm{u}_v^i, \mathrm{u}_v^{i*}$ - solutions of the equations (1) and (7) with $\underline{\eta}^1, i = 1,2$ fixed. Then the solution of the inverse problem (1), (2) on set $\mathcal{M}_{\underline{\eta}_0}^c$ exists, the only and any minimizing sequence $\left\{\underline{\eta}^j\right\} \in \mathcal{M}_{\underline{\eta}_0}^c$, $j = 1,2...$, is weakly convergent to it irrespective of the initial estimate.

The proof is based on lemmas 1,2 and laying down great convexity of the functional (3) on set $\mathcal{M}_{\underline{\eta}_0}^c$.

Remark 2. The set $\mathcal{M}_{\underline{\eta}_0}^c$ is referred to as correctness set in the theory of conditionally correct problems. The parameter c has the meaning of radius of this set.

Definition 2. Let Y_m be the higher of the correctnesses sets $\mathcal{M}_{\underline{\eta}_0}^c$, $c > 0$, on each of them lemmas 1-3 conditions are satisfied for m integrative methods. We will call the set Y_m limit correctness set for the indexed methods $v = 1,...,m$.

Theorem 1. Let $\{Y_m\}$, $m = 1,...,M$, be the complex of limit sets of correctnesses for M integral Geophysics inverse problems (1), (2), all of which are based on integration of m

methods $(v=1,...,m)$. Then we have embedding of sets Y_m of the type $Y_1 \subset Y_2 \subset ... \subset Y_M$, along with this the radius ρ_κ of κ limit set of correctness is defined by the recurrence relation: $\rho_1 = s_1, \rho_\kappa = \left(\rho_{\kappa-1}^2 + s_\kappa^2\right), \quad \kappa = 2,...,M$, where κ is the radius of correctness of a particular κ method.

The proof follows from the lemmas 1-3.

Let us consider the issues of accuracy and stability of solution of the complex inverse problem (1), (2). The accuracy of determination of initial parameters is known to be characterized by the value of variance of their estimate. Let σ_ι be the variance of estimate l of component of the vector $\underline{\eta} = \left(\underline{\eta}_1,...,\underline{\eta}_L\right)$ in integral statement of the inverse problem, σ_{vl} - a similar estimate which is obtained only by solution of the problem using the method with v, σ_l^0 index – square root of l diagonal element of the matrix which is inverse to the covariance matrix $G_{\underline{\eta}}$. Stability of inverse problem solution is known to be dependent on «ravine» of the objective functional which in turn is characterized by Fisher matrix condition exponent [12]. The less is this exponent, the less is «ravine», that is the higher is stability and minimal solution accuracy $\left(\min_l \{\sigma_l\}\right)$. Let ζ be the condition exponent of the matrix $G_{\underline{\eta}}^{-1}, \xi_v$ - is the condition exponent of Fisher matrix of v method, ξ - integral statement condition exponent of Fisher matrix.

Theorem 2. Solution of the integral inverse problem (1), (2) on the set of correctness $\mathcal{M}_{\underline{\eta}_0}^c$ is characterized by the estimates of accuracy and stability as follows:

$$\sigma_l^2 \le 2^{2^{(1-m)}} \max_{v=1,m} \{\sigma_{vl}\} \sigma_l^0, l = 1..., L,$$
$$\xi \le \max\{\zeta, \psi\}, \psi \le \max_{v=1,m} \{\xi_v\}$$

The proof is based on lemmas 1,2 and properties of symmetrical positively defined matrices.

Remark 3. When estimating stability (the case when stability and accuracy do not increase in integration), the equality is possible only if the following three conditions are simultaneously fulfilled: there is no correlative connection between the target parameters of vector $\underline{\eta}$ (the matrix $G_{\underline{\eta}}$ is diagonal); there are no crossed parameters in mathematical models used; there are no univalently solvable into vector components d_v determinate constraint equations.

Remark 4. According to the estimated variance σ_l, if the amount of methods is increased, integral accuracy of parameters determination improves as a whole. However, direct use of the method with big own variance σ_{vl}, in spite of the fact that it makes negligible contribution to the functional (3), may spoil the overall picture. To avoid this, when numerical calculation is being carried out, it is apparently necessary to invoke other aprior information – experience of specialists on integral interpretation of particular geological features. Such technology for quantitative solution of integral Geophysics problems is possible on the basis of hybrid expert systems use.

Computation center
The Siberian Branch of the USSR Academy of Sciences
Novosibirsk
Received
24/08/1988

References

1. P.S. Novikov. 1938. *Academy of Science Proceedings*. V. 18. 165-168.
2. A.N. Tikhonov. 1943. *Academy of Sciences Proceedings*. V. 39. №5. 195-198.
3. A.S. Alekseev. 1962. *USSR Academy of Science Review*. Geophysics series. №11. 1514-1531.
4. A.S. Alekseev. 1967. *Some Methods and Algorithms of Geophysical Data Interpretation. Moscow*. «Science» Press. 9-84.
5. A.S. Blagoveshchensky. 1966. *Mathematical Geophysics Problems*. Leningrad. Leningrad State University Press. V. 1. 68-81.
6. V.G. Romanov. 1980. *Mathematical Geophysics Inverse Problems*. Moscow. «Science» Press.
7. V.N. Strakhov. 1962. *USSR Academy of Science Review*. Geophysics series. № 3/4. 9; 307-316; 336-347; 491-507.
8. A.S. Alekseev, B.A. Bubnov. 1981. *Academy of Science Proceedings*. V. 261. № 5. 1086-1090.
9. G.Ya. Golizdra. 1978. *USSR Academy of Science Review*. Geophysics Series. № 6. 26-38.
10. Yu.E. Anikonov, G.N. Erokhin. 1985. *Research Methods for Nonclassical Problems of Mathematical Geophysics*. Novosibirsk. 55-63.
11. R.N. Hodgson, R.C. Ross. 1983. *Oil and Gas*. №5. 115-126.
12. F.N. Goltsman, T.Yu. Kalinina. 1983. *Statistical Interpretation of Magnetic and Gravitation Anomalies*. Leningrad. «Mineral Resources» Press.
13. A.S. Alekseev, G.M. Tsybulchik. 1978. *Academy of Science Proceedings*. V. 242. № 5. 1030-1033.
14. B.Ya. Zeldovich, V.V. Shkunov. 1986. *In the World of Science*. № 2. 16-24.
15. G.I. Marchuk. 1976. *Academy of Science Proceedings*. V. 227, № 5. 1056-1059.